THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Suggested Solution for Quiz 1

1. (a) Note that $\emptyset, \mathbb{R} \in \mathfrak{T}$.

   Pick any arbitrary collection of elements $U_\alpha \in \mathfrak{T}$. If at least one of U_α is $\mathbb{R}$, then we have $\cup_{\alpha \in I} U_\alpha = \mathbb{R} \in \mathfrak{T}$. Otherwise, WLOG we assume that for all $\alpha \in I$, $U_\alpha = (n_\alpha, \infty)$ for some $n_\alpha \in \mathbb{N}$. Then $\cup_{\alpha \in I} U_\alpha = (n, \infty) \in \mathfrak{T}$, where $n = \min\{n_\alpha \mid \alpha \in I\}$. Hence arbitrary union of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$.

   Pick any $U_1, U_2, \ldots, U_k \in \mathfrak{T}$. If at least one of U_i is $\emptyset$, then $\cap_{i=1}^k U_i = \emptyset \in \mathfrak{T}$. Otherwise, WLOG we assume that for all $i = 1, 2, \ldots, k$, $U_i = (n_i, \infty)$ for some $n_i \in \mathbb{N}$. Then we have $\cap_{i=1}^k U_i = (n, \infty) \in \mathfrak{T}$, where $n = \max\{n_i \mid i = 1, 2, \ldots k\}$. Hence arbitrary intersection of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$.

   As a result, $\{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{N}\}$ is a topology for $\mathbb{R}$.

   (b) By dense property of rational number, pick a decreasing sequence $q_n \in \mathbb{Q}$ which converges to $\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q}$. Then we have $\cup_{i=1}^\infty (q_n, \infty) = (\sqrt{2}, \infty) \notin \mathfrak{T}$. Hence $\{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{Q}\}$ is not a topology for $\mathbb{R}$.

   (c) Note that $\emptyset, \mathbb{R} \in \mathfrak{T}$.

   Pick any arbitrary collection of elements $U_\alpha \in \mathfrak{T}$. If at least one of U_α is $\mathbb{R}$, then we have $\cup_{\alpha \in I} U_\alpha = \mathbb{R} \in \mathfrak{T}$. Otherwise, WLOG we assume that for all $\alpha \in I$, $U_\alpha = (r_\alpha, \infty)$ for some $r_\alpha \in \mathbb{R}$. If $\{r_\alpha\}$ is bounded below, then $\cup_{\alpha \in I} U_\alpha = (r, \infty) \in \mathfrak{T}$, where $n = \inf\{r_\alpha \mid \alpha \in I\}$. Otherwise, we have $\cup_{\alpha \in I} U_\alpha = \mathbb{R} \in \mathfrak{T}$. Hence arbitrary union of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$.

   Pick any $U_1, U_2, \ldots, U_k \in \mathfrak{T}$. If at least one of U_i is $\emptyset$, then $\cap_{i=1}^k U_i = \emptyset \in \mathfrak{T}$. Otherwise, WLOG we assume that for all $i = 1, 2, \ldots, k$, $U_i = (r_i, \infty)$ for some $n_i \in \mathbb{N}$. Then we have $\cap_{i=1}^k U_i = (r, \infty) \in \mathfrak{T}$, where $r = \max\{r_i \mid i = 1, 2, \ldots k\}$. Hence arbitrary intersection of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$.

   As a result, $\{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ is a topology for $\mathbb{R}$.

2. (a) First let us show that $X \backslash \operatorname{Int}(A) = \overline{X \backslash A}$. Pick $x \in X \backslash \operatorname{Int}(A)$. Then $x \notin \operatorname{Int}(A)$. So for any $U \in \mathfrak{T}$ with $x \in U$, we have $U \not\subset A$ and hence $U \cap (X \backslash A) \neq \emptyset$. Hence $x \in \overline{X \backslash A}$ and $X \backslash \operatorname{Int}(A) \subset \overline{X \backslash A}$. Pick $x \in \overline{X \backslash A}$. Then for any $U \in \mathfrak{T}$ with $x \in U$, we have $U \cap (X \backslash A) \neq \emptyset$. Hence $U \not\subset A$ and hence $x \in X \backslash \operatorname{Int}(A)$. This shows that $X \backslash \operatorname{Int}(A) = \overline{X \backslash A}$. Thus we have

$$X \backslash (\overline{X \backslash A}) = X \backslash (X \backslash \operatorname{Int}(A)) = \operatorname{Int}(A)$$

   (b) Let $A = \mathbb{Q}$ and $B = \mathbb{R} \backslash \mathbb{Q}$ in $(\mathbb{R}, \mathfrak{T}_{\text{std}})$. Then we have $\operatorname{Int}(A) = \operatorname{Int}(B) = \emptyset$ and $\operatorname{Int}(A \cup B) = \operatorname{Int}(\mathbb{R}) = \mathbb{R}$. Hence $\operatorname{Int}(A \cup B) \neq \operatorname{Int}(A) \cup \operatorname{Int}(B)$ in general.

3. Let $\mathfrak{T}_{cfA} = \mathfrak{T}_1 \cup \mathfrak{T}_2$, where $\mathfrak{T}_1 = \{G \subset X : G \cap A = \emptyset\}$ and $\mathfrak{T}_2 = \{G \subset X : A \subset G, X \backslash G \text{ is finite}\}$. Since $\emptyset \cap A = \emptyset$, we have $\emptyset \in \mathfrak{T}_{cfA}$. Also, since $A \subset X$ and $X \backslash X = \emptyset$ is finite, we have $X \in \mathfrak{T}_{cfA}$.

   Pick any arbitrary collection of elements $U_\alpha \in \mathfrak{T}_{cfA}$. If $U_\alpha \in \mathfrak{T}_1$ for all α, then $(\cup_{\alpha \in I} U_\alpha) \cap A = \cup_{\alpha \in I} (U_\alpha \cap A) = \emptyset$ and hence $\cup_{\alpha \in I} U_\alpha \in \mathfrak{T}_1 \subset \mathfrak{T}_{cfA}$. Otherwise, we have $U_o \in \mathfrak{T}_2$ for some $o \in I$. Then we have $A \subset U_o \subset \cup_{\alpha \in I} U_\alpha$ and $X \backslash (\cup_{\alpha \in I} U_\alpha) \subset X \backslash U_o$ is finite. Hence $\cup_{\alpha \in I} U_\alpha \in \mathfrak{T}_2 \subset \mathfrak{T}_{cfA}$.

Pick any $U_1, U_2, \ldots, U_k \in \mathfrak{T}_{cfA}$. If there exists m such that $U_m \in \mathfrak{T}_1$, then $(\cap_{i=1}^k U_i) \cap A \subset U_m \cap A = \emptyset$. Hence $(\cap_{i=1}^k U_i) \in \mathfrak{T}_1 \subset \mathfrak{T}_{cfA}$. Otherwise, we have $U_i \in \mathfrak{T}_2$ for all i. Since $A \subset U_i$ for all i, we have $A \subset \cap_{i=1}^k U_i$. Furthermore, the set $X \backslash (\cap_{i=1}^k U_i) = \cup_{i=1}^k X \backslash U_i$, being a finite union of finite set, is finite. Hence $\cap_{i=1}^k U_i \in \mathfrak{T}_2 \subset \mathfrak{T}_{cfA}$.

As a result, $\mathfrak{T}_{cfA}$ is a topology for X.

4. (a) The statement is true. Recall that $\mathfrak{T}_{std} \subset \mathfrak{T}_{ll}$. Given a continuous function $f : \mathbb{R}_{std} \to \mathbb{R}_{ll}$. Pick any open set $U \in \mathfrak{T}_{std}$. Then we have $U \in \mathfrak{T}_{ll}$. By continuity, we have $f^{-1}(U) \in \mathfrak{T}_{std}$. Hence f is a continuous function from $\mathbb{R}_{std}$ to $\mathbb{R}_{std}$.

   (b) The statement is false. Consider the function $f : \mathbb{R}_{ll} \to \mathbb{R}_{std}$ defined by $f(x) = 0$ for all $x < 0$ and $f(x) = 1$ for all $x \geq 0$. Note that for any open set U, $f^{-1}(U)$ is equal to (i) $\emptyset$; (ii) $[0, \infty)$; (iii) $(-\infty, 0)$ or (iv) $\mathbb{R}$. Hence f is continuous. However it is not continuous as a function from $\mathbb{R}_{std}$ to $\mathbb{R}_{std}$.

   (c) Same as (b).

   (d) Consider the function $f(x) = -x$. Clearly $f : \mathbb{R}_{std} \to \mathbb{R}_{std}$ is continuous. However, we have $f^{-1}([0, 1)) = (-1, 0] \notin \mathfrak{T}_{ll}$. Hence f is not a continuous function form $\mathbb{R}_{ll}$ to $\mathbb{R}_{ll}$.

5. (a) The statement is true. Consider the countable set $\mathbb{Z} \subset \mathbb{R}$. Pick any open set U. By definition of cofinite topology, we know that $X \backslash U$ is finite. If $U \cap \mathbb{Z} = \emptyset$, we have $\mathbb{Z} \subset X \backslash U$, contradicting the fact that $X \backslash U$ is finite. Hence $U \cap \mathbb{Z} \neq \emptyset$ and $\mathbb{Z}$ is a countable dense subset in $(\mathbb{R}, \mathfrak{T}_{cf})$.

   (b) The statement is false. See Tutorial classwork 1 Q1)a).

   (c) The statement is false. See Tutorial classwork 0 Q1)b).

6. Assume that A' is countable. Then $A \backslash A'$ is uncountable. For each $x \in A \backslash A'$, we can find $B_x \in \mathfrak{B}$ such that $x \in B_x$ and $B_x \cap A \backslash \{x\} = \emptyset$. In particular, for any $x, y \in A \backslash A'$ with $x \neq y$, we must have $B_x \neq B_y$, otherwise we have $y \in B_x \cap A \backslash \{x\}$, contradiction. Hence $B_x \neq B_y$ for any $x \neq y$. This gives us an injective map from $A \backslash A'$ to $\mathfrak{B}$. However, since the set $A \backslash A'$ is uncountable while the set $\mathfrak{B}$ is countable, such mapping cannot exist. This leads to contradiction. Hence A' must be uncountable.